

# Well-orders in the transfinite Japaridze algebra

David Fernández-Duque  
 Group for Computational Logic  
 Universidad de Sevilla  
 davidstofeles@gmail.com  
<http://personal.us.es/dfduque/indexen.html>

Joost J. Joosten  
 Department of Logic, History and Philosophy of Science,  
 University of Barcelona  
 jjoosten@ub.edu  
<http://www.phil.uu.nl/~jjoosten/>

Keywords: Provability logics, well-orders,  
 Japaridze algebra, worms, hyperexponentials

December 17, 2012

## Abstract

The logic GLP is a polymodal logic that has for each ordinal  $\alpha$  an operator  $[\alpha]$ , whose intended interpretation is a provability predicate in a hierarchy of theories of increasing strength. Its corresponding algebra is called the (transfinite) Japaridze algebra. There are various natural orders in this algebra that are based on comparing consistency strength of its elements. In particular, for each  $\alpha$  we define  $A <_\alpha B \Leftrightarrow \text{GLP} \vdash B \rightarrow \langle \alpha \rangle A$ .

In this paper we shall consider *worms*, which are formulas of the form  $\langle \alpha_0 \rangle \dots \langle \alpha_n \rangle \top$ , and the partial orders  $<_\alpha$  on their images in the Japaridze algebra. By  $S_\alpha$  we will denote the class of worms that only contain ordinals at least as big as  $\alpha$ . Given an ordinal  $\alpha$  and a worm  $A \in S_\alpha$ , a major goal is to show how one computes the order type of  $\{B \in S_\alpha : B <_\alpha A\}$ .

Our main results show how these order types can be computed via hyperexponentials which are forms of transfinite iterations of ordinal functions closely related to Veblen hierarchies. These novel methods set this paper apart from a previously published study ([3]) to determine the same order types.

# 1 Introduction

Gödel-Löb Polymodal logic,  $\text{GLP}$  for short, is a provability logic that has for each ordinal  $\alpha$  a modality  $[\alpha]$ . The idea is that for  $\alpha < \beta$ , the operator  $[\beta]$  represents provability in a theory stronger than that corresponding to  $[\alpha]$ , so that  $[\alpha]\psi \rightarrow [\beta]\psi$  holds. We shall write  $\text{GLP}_\Lambda$  for the part of  $\text{GLP}$  where one only has modalities  $[\alpha]$  for  $\alpha < \Lambda$ .

The logic  $\text{GLP}_\omega$  was first introduced by Japaridze in [9], where  $[n]$  was read as “provable with  $n$  applications of the  $\omega$ -rule”. Later, Ignatiev studied  $\text{GLP}_\omega$  in more detail in [8], showing it arithmetically complete for other readings too. For the current paper the actual (hyper)arithmetical reading of the modalities is not important and we study the logics from a purely modal and algebraic perspective.

Lately, interest in the logics  $\text{GLP}_\Lambda$  has revived since Beklemishev applied  $\text{GLP}_\omega$  to give a  $\Pi_1^0$  ordinal analysis of Peano Arithmetic, various of its sub-theories and some simple extensions (see [2]). This paradigm of  $\Pi_1^0$  ordinal analysis looks very promising as it is more fine-grained than other proof theoretic ordinals ( $\Pi_2^0$ ,  $\Pi_1^1$ ) and can distinguish the proof theoretic strength of, for example,  $\text{PA}$  and  $\text{PA} + \text{Con}(\text{PA})$  where the others cannot.

However, the theories which allow for a  $\Pi_1^0$  ordinal analysis so far have not been very strong. For theories considerably stronger than  $\text{PA}$  the machinery needs to be enhanced. A first step in this direction was pursued in [3] where the logics  $\text{GLP}_\Lambda$  were introduced. This paper can be seen as a natural continuation of [3] in that we study and analyze the orders introduced there in a more general setting.

All orderings we shall introduce are important in the general project of applying  $\text{GLP}_\Lambda$  to ordinal analysis: they are closely related to the ordinal representation systems and the meta-mathematical properties of the progressions that arise when transfinitely iterating consistency assertions over some base theory.

## 1.1 The logics $\text{GLP}_\Lambda$

In the definition below the  $\alpha$  and  $\beta$  range over ordinals and the  $\psi$  and  $\chi$  over formulas in the language of  $\text{GLP}_\Lambda$ . The language of  $\text{GLP}_\Lambda$  is that of propositional modal logic that contains for each  $\alpha < \Lambda$  a unary modal operator  $[\alpha]$ .

**Definition 1.1.** *For  $\Lambda$  an ordinal, the logic  $\text{GLP}_\Lambda$  is the propositional normal modal logic that has for each  $\alpha < \Lambda$  a modality  $[\alpha]$  and is axiomatized by the following schemata:*

$$\begin{aligned} &[\alpha](\chi \rightarrow \psi) \rightarrow ([\alpha]\chi \rightarrow [\alpha]\psi), \\ &[\alpha]([\alpha]\chi \rightarrow \chi) \rightarrow [\alpha]\chi, \\ &\langle \alpha \rangle \psi \rightarrow [\beta] \langle \alpha \rangle \psi && \text{for } \alpha < \beta, \\ &[\alpha]\psi \rightarrow [\beta]\psi && \text{for } \alpha \leq \beta. \end{aligned}$$

*The rules of inference are Modus Ponens and necessitation for each modality:  $\frac{\psi}{[\alpha]\psi}$ . By  $\text{GLP}$  we denote the class-size logic that has a modality  $[\alpha]$  for each ordinal  $\alpha$  and all the corresponding axioms and rules.*

It is good to recall that from Löb's axiom  $[\alpha]([\alpha]\chi \rightarrow \chi) \rightarrow [\alpha]\chi$  one can easily derive transitivity, that is,

$$[\alpha]\chi \rightarrow [\alpha][\alpha]\chi,$$

and we shall use this freely in our reasoning.

## 1.2 Worms and the closed fragment of GLP

It turns out that most calculations needed for a  $\Pi_1^0$ -ordinal analysis can be performed in the *closed fragment* of GLP. A closed formula in the language of GLP is simply a formula without propositional variables. In other words, closed formulas are generated by just  $\top$  and the Boolean and modal operators.

The closed fragment of GLP is just the set of closed formulas provable in GLP. Within this closed fragment and the corresponding algebra, there is a particular class of privileged inhabitants/terms which are called *worms*.

**Definition 1.2** (Worms,  $S$ ,  $S_\alpha$ ). *By  $S$  we denote the set of worms of GLP which is inductively defined as  $\top \in S$  and  $A \in S \Rightarrow \langle \alpha \rangle A \in S$ . Similarly, we inductively define for each ordinal  $\alpha$  the set of worms  $S_\alpha$  where all ordinals are at least  $\alpha$  as  $\top \in S_\alpha$  and  $A \in S_\alpha \wedge \beta \geq \alpha \Rightarrow \langle \beta \rangle A \in S$ .*

Both the closed fragment of GLP and the set of worms have been studied to quite some extent in [3] and [1]. It is known that each closed formula of GLP can be written as a Boolean combination of worms and that the closed fragment of  $\text{GLP}_\Lambda$  is decidable for decidable  $\Lambda$ .

We shall identify a worm  $A$  in the obvious way with  $\iota(A)$ , the string of ordinals in the consistency statements that is involved in  $A$ :  $\iota(\top) = \lambda$  and  $\iota(\langle \alpha \rangle A) = \alpha * \iota(A)$ . In this paper  $\lambda$  will denote the empty string. Worms can thus be perceived as strings over the ordinals and for this reason are also sometimes called *words*. We call them worms here as to refer to the heroic worm-battle, a variant of the Hydra battle (see [4]).

Apart from identifying a worm with its corresponding string of ordinals we shall use any hybrid combination in between at times. For example, we might equally well write  $\omega 0 \omega$ , as  $\langle \omega \rangle 0 \omega$ , or  $\langle \omega \rangle \langle 0 \rangle \langle \omega \rangle \top$ .

The following lemma follows easily from the axioms of GLP and shall be used repeatedly without explicit mention in the remainder of this paper.

**Lemma 1.3.** *In this lemma,  $A$ ,  $B$  and  $C$  denote worms and  $\alpha$  and  $\beta$  denote ordinals.*

1. *For a GLP formula  $\phi$  and a worm  $B$ , if  $\beta < \alpha$ , then*  

$$\text{GLP} \vdash (\langle \alpha \rangle \phi \wedge \langle \beta \rangle B) \leftrightarrow \langle \alpha \rangle (\phi \wedge \langle \beta \rangle B);$$
2. *If  $A \in S_{\alpha+1}$ , then  $\text{GLP} \vdash A \wedge \langle \alpha \rangle B \leftrightarrow A \alpha B$ ;*
3. *If  $A, B \in S_\alpha$  and  $\text{GLP} \vdash A \leftrightarrow B$ , then*  

$$\text{GLP} \vdash A \alpha C \leftrightarrow B \alpha C.$$

*Proof.* The  $\rightarrow$  direction of the first item follows from the axiom  $\langle\beta\rangle B \rightarrow [\alpha]\langle\beta\rangle B$ . For the other direction we observe that  $\langle\alpha\rangle\langle\beta\rangle B \rightarrow \langle\beta\rangle B$  in virtue of axiom  $\langle\alpha\rangle\langle\beta\rangle B \rightarrow \langle\beta\rangle\langle\beta\rangle B$  and transitivity of  $[\beta]$ . The other two items follow directly from the first.  $\square$

### 1.3 Plan of the paper

After the introduction, in Section 2 we will revisit some standard notions from ordinal arithmetic that are needed throughout the rest of the paper.

In Section 3 we introduce the linear orders  $<_\alpha$  on  $S_\alpha$  defined as  $A <_\alpha B :\Leftrightarrow \text{GLP} \vdash B \rightarrow \langle\alpha\rangle A$ . The function  $o$  will map a worm to the order type of the set  $\{B \in S \mid B <_0 A\}$ . Likewise, the functions  $o_\alpha$  will map a worm  $A \in S_\alpha$  to the order type of the set  $\{B \in S_\alpha \mid B <_\alpha A\}$ .

In Section 4 a calculus is given for  $o$ . The calculus reduces the computation of  $o$  to a family of functions  $e^\alpha$  which are defined as the functions that enumerate the respective classes  $o(S_\alpha)$ . We call these functions  $e^\alpha$  hyperexponentiation.

In Section 5 basic properties of the hyperexponentiations  $e^\alpha$  are proven and a full recursive scheme is given for computing them.

Finally, in Section 6, it is proven that  $e^\alpha$  can be seen as a transfinite iterate that we call *hyperation*. In particular, this provides us with a link to well-known Veblen progressions.

Various results in this paper were also stated in [3]. The current presentation is substantially different though, and mostly self-contained. In particular, the calculus that we present is of a different nature than the one presented in [3] and the methods by which we obtain the calculus are different too. In the last section we shall see that both calculi yield the same values as is to be expected.

This paper is the first in a series of two. In the current paper we study the sets  $\{B \in S_\alpha \mid B <_\alpha A\}$  for  $A \in S_\alpha$ . In the follow-up to this paper (*Well-orders in the transfinite Japaridze algebra II*, [7]) and in [6] we shall see that making just a minor change to this set causes a drastic change. If we consider  $\{B \in S_0 \mid B <_\alpha A\}$  for  $A \in S_0$  we see that the set, although still well-founded, is no longer linearly ordered by  $<_\alpha$ . In particular, we shall see that the set is much more wildly behaved and contains infinite anti-chains.

### 1.4 Notation

We reserve lower-case Greek letters  $\alpha, \beta, \gamma, \dots, \xi, \dots$  for variables ranging over ordinals. Worms will be denoted by upper case latin letters  $A, B, C, \dots$ . The Greek lower-case letters  $\phi, \psi, \chi, \dots$  will denote formulas. However,  $\varphi$  shall be reserved for the Veblen enumeration function and variants thereof. Likewise, we reserve  $\omega$  to denote the first infinite ordinal.

## 2 Ordinal arithmetic

As we shall study well-orders, we need some ordinal arithmetic. In this section we shall just state without proof the main properties that we need. For further definitions and detailed proofs, we refer the reader to [10]. Ordinals are canonical representatives for well-orders. The first infinite ordinal is as always denoted by  $\omega$ .

Most operations on natural numbers can be extended to ordinal numbers, like addition, multiplication and exponentiation (see [10]). However, in the realm of ordinal arithmetic things become often more subtle. For example  $1 + \omega = \omega \neq \omega + 1$  and also the other operations differ considerably from ordinary arithmetic.

However, there are various similarities too. In particular we have a form of subtraction available in ordinal arithmetic.

**Lemma 2.1.**

1.  $\forall \zeta < \xi \exists! \eta \zeta + \eta = \xi$   
(We will denote this unique  $\eta$  by  $-\zeta + \xi$ ),
2.  $\forall \eta > 0 \exists! \alpha, \beta \eta = \omega^\alpha + \beta$  such that  $\beta < \omega^\alpha + \beta$ .

One of the most useful ways to represent ordinals is through their Cantor Normal Forms (CNFs):

**Theorem 2.2** (Cantor Normal Form Theorem).

For each ordinal  $\alpha$  there are unique ordinals  $\alpha_1 \geq \dots \geq \alpha_n$  such that

$$\alpha = \omega^{\alpha_1} + \dots + \omega^{\alpha_n}.$$

We call a function  $f$  *increasing* if  $\alpha < \beta$  implies  $f(\alpha) < f(\beta)$ . An ordinal function is called *continuous* if  $\bigcup_{\zeta < \xi} f(\zeta) = f(\xi)$  for all limit ordinals  $\xi$ . Functions which are both increasing and continuous are called *normal*.

## 3 Linear orders on the Japaridze algebra

In this section we shall introduce linear orders on worms and state some basic known facts about them.

### 3.1 The orderings $<_\alpha$

It is known ([3, 1]) that the class of worms is linearly ordered by consistency strength. That is, two worms are either equivalent or one of the two implies the consistency (0-consistency that is) of the other. This ordering is a main theme of this paper.

**Definition 3.1** ( $<, <_\alpha, o, o_\alpha$ ). We define a relation  $<_\alpha$  on  $S_\alpha \times S_\alpha$  by

$$A <_\alpha B \iff \text{GLP} \vdash B \rightarrow \langle \alpha \rangle A \quad (\text{with } A, B \in S_\alpha).$$

For  $A \in S_\alpha$  we denote by  $o_\alpha(A)$  the order type of  $\{B \in S_\alpha \mid B <_\alpha A\}$ . More precisely, for  $A \in S_\alpha$  we define inductively

$$o_\alpha(A) = \sup \{o_\alpha(B) + 1 : B \in S_\alpha \text{ \& } B <_\alpha A\},$$

where  $\sup \emptyset = 0$ .

When  $X$  is a set or class we shall denote by  $o_\alpha(X)$  the image of  $X$  under  $o_\alpha$ .

Instead of  $<_0$  and  $o_0$  we shall write  $<$  and  $o$ , respectively. The restriction to  $S_\alpha$  in the definition of  $o_\alpha$  is essential. In [7, 6] the orders that arise when dropping this restriction are studied and they turn out to be of a completely different nature.

### 3.2 Japaridze algebras

The relations  $<_\alpha$  do not give proper linear orders on  $S_\alpha$ , given that different worms may be equivalent in GLP and hence undistinguishable in the ordering. We remedy this by passing to the Lindenbaum algebra of GLP – that is, the quotient of the language of GLP modulo provable equivalence.

This algebra is a *Japaridze algebra*, as described below:

**Definition 3.2** (Japaridze algebra). A Japaridze algebra is a structure

$$\mathcal{J} = \langle D, \{[\alpha]\}_{\alpha < \Lambda}, \wedge, \neg, 0, 1 \rangle$$

such that

1.  $\langle D, \wedge, \neg, 0, 1 \rangle$  is a Boolean algebra,
2.  $[\alpha]1 = 1$  for all  $\alpha < \Lambda$ ,
3.  $[\alpha](x \rightarrow y) \leq [\alpha]x \rightarrow [\alpha]y$  for all  $\alpha < \Lambda$ ,  $x, y \in D$ ,
4.  $[\alpha]([\alpha]x \rightarrow x) \leq [\alpha]x$  for all  $\alpha < \Lambda$ ,  $x \in D$ ,
5.  $[\alpha]x \leq [\beta]x$  for all  $\alpha \leq \beta < \Lambda$ ,  $x \in D$  and,
6.  $\langle \alpha \rangle x \leq [\beta] \langle \alpha \rangle x$  for all  $\alpha < \beta < \Lambda$ ,  $x \in D$ ,

where  $\langle \alpha \rangle, \rightarrow$  are defined in the usual way.

It is in these algebras that the partial orders  $<_\alpha$  we have described naturally reside. However, rather than work with abstract elements of the Lindenbaum algebra, it will be convenient to choose suitable representatives of each equivalence class. These representatives will be given by *Beklemishev Normal Forms* (BNFs), as described below.

### 3.3 A well-order on Beklemishev Normal Forms

Beklemishev Normal Forms are a subclass of  $S$  where  $<_0$  does define a linear order as was shown in [1, 3]. Also it is shown there that each worm is equivalent to a unique worm in BNF and that this BNF can be found effectively for recursive well-orders. Moreover, if  $A \in S_\alpha$ , then its equivalent in BNF is also in  $S_\alpha$ .

In the next section we shall provide a calculus to compute  $o_\alpha$ . Note that it is not at all obvious that  $o_\alpha$  is defined everywhere, but this turns out to be the case.

It is easy to see that  $<_\alpha$  is transitive. In [1] the relation was shown to be irreflexive and in [3] it was shown to be well-founded on  $S_\alpha$  if irreflexive. Thus indeed,  $o_\alpha$  is well-defined.

**Definition 3.3** (Beklemishev Normal Form). *A worm  $A \in S$  is in BNF (Beklemishev Normal Form) iff*

1.  $A = \lambda$  *or,*
2.  $A$  is of the form  $A_k \alpha \dots \alpha A_1$  with  $\alpha = \min(A)$ ,  $k \geq 1$  and  $A_i \in S_{\alpha+1}$  such that each  $A_i$  is in BNF and moreover  $A_{i+1} \leq_{\alpha+1} A_i$  for each  $i < k$ .

We shall write  $\mathcal{B}$  for BNF and  $\mathcal{B}_\alpha$  for  $\text{BNF} \cap S_\alpha$ .

**Lemma 3.4.** *Each worm of the form  $\alpha^n$ , i.e.,  $\overbrace{\langle \alpha \rangle \dots \langle \alpha \rangle}^{n \text{ times}} \top$ , is in BNF.*

*Proof.* This is immediate if we conceive  $\alpha^n$  as  $\lambda \alpha \lambda \dots \lambda \alpha \lambda$ . □

Definition 3.3 already reveals a strong similarity between BNFs and CNFs. The similarity is even further stressed by Lemmas 3.7 and 3.9 below. We first need some definitions.

**Definition 3.5.** *For  $(X, \prec)$  a well-order we will denote its order type by  $\text{ot}(X, \prec)$ . Similarly, for  $a \in X$  we denote the order-type of*

$$X(a) = \{b : b \prec a\}$$

*under  $\prec \upharpoonright X(a)$  by  $\text{ot}(a, \prec)$ .*

**Definition 3.6.** *Let  $(X, \prec)$  be a well-order. By  $(X^{<\omega}, \prec^L)$  we denote the lexicographical ordering on the finite sequences of elements of  $X$ . Thus, the empty sequence is  $\prec^L$ -below any non-empty sequence and  $(x_1, x_2, \dots, x_n) \prec^L (y_1, y_2, \dots, y_m)$  iff*

1.  $x_1 \prec y_1$  *or*
2.  $x_1 = y_1$  and  $(x_2, \dots, x_n) \prec^L (y_2, \dots, y_m)$ .

It is well-known that if  $(X, \prec)$  is a well-order then so is  $(X^{<\omega}, \prec^L)$ . The following lemma is folklore:

**Lemma 3.7.** *Let  $(X, \prec)$  be a well-order with  $\text{ot}(X, \prec) = \alpha$ . Then,*

$$\text{ot}(X^{<\omega}, \prec^L) = \omega^\alpha$$

*and for each sequence  $(x_1, \dots, x_n)$  in  $X^{<\omega}$  we have*

$$\text{ot}((x_1, \dots, x_n), \prec^L) = \omega^{\text{ot}(x_1, \prec)} + \dots + \omega^{\text{ot}(x_n, \prec)}.$$

In order to concisely formulate certain facts, let us first introduce some notation.

**Notation 3.8.**

$$A_k \alpha \dots A_1 := \begin{cases} \lambda & \text{for } k = 0, \\ A_1 & \text{for } k = 1, \\ A_k \alpha (A_{k-1} \alpha \dots A_1) & \text{otherwise.} \end{cases}$$

The next lemma is proven in both [3] and [1]. It tells us that the  $<_\alpha$ -order on  $S_\alpha$  can be conceived as a lexicographical construct over the  $<_{\alpha+1}$ -order on  $S_{\alpha+1}$ .

**Lemma 3.9.** *Consider two worms  $A = A_m \alpha \dots A_1$  and  $B = B_n \alpha \dots B_1$  both in  $\mathcal{B}_\alpha$  with  $A_i, B_j \in \mathcal{B}_{\alpha+1}$ , and not both  $A_1$  and  $B_1$  empty. We have that*

$$A <_\alpha B \Leftrightarrow (A_1, \dots, A_m) <_{\alpha+1}^L (B_1, \dots, B_n).$$

## 4 A calculus for $o$

In this section we prove the basic properties of  $o$  that we need so that a calculus for computing  $o$  and  $o_\alpha$  can be obtained. However, the calculus that is provided in this section is formulated using a collection of ordinal functions  $\{e^\alpha \mid \alpha \in \text{Ord}\}$ . In Section 5 we shall see how these functions can be computed.

### 4.1 Basic properties

The following lemma contains some basic observations.

**Lemma 4.1.** *Let  $A$  be any worm.*

1.  $o(\lambda) = 0$ ,
2.  $o(0A) = o(A) + 1$ ,
3. *If  $A = A_k 0 \dots A_1 \in \mathcal{B}$  with all  $A_i \in S_1$  and  $A_1$  not empty, then  $o(A) = \omega^{o_1(A_1)} + \dots + \omega^{o_1(A_k)}$ .*

*Proof.* The first item is clear as by the irreflexivity of  $<$ ,  $\text{GLP} \not\vdash \top \rightarrow \Diamond A$  for any worm  $A$ .

For the second item, assume that for some  $B$  we have  $A < B < 0A$ . Then  $0A \rightarrow 0B$  is derivable, but as  $B \rightarrow 0A$  this would yield  $0A \rightarrow 00A$  contradicting irreflexivity.

The third item follows directly from Lemmas 3.7 and 3.9. □



As mentioned above,  $<_\alpha$  defines a well-order on  $\mathcal{B}_\alpha$ . In particular the class of all worms in BNF is well-ordered by  $<_0$ . *Par abus de langage* we will denote the restriction of  $o$  to  $\mathcal{B}$  also by  $o$ . The following lemma is standard. In virtue of the definition of  $o$  the proof can be found in any textbook that explains well-orders. We give a proof in terms of the worms themselves.

**Lemma 4.2.** *The map  $o : (\mathcal{B}, <_0) \rightarrow (\text{Ord}, <)$  defines an isomorphism.*

*Proof.* For  $A, B \in \mathcal{B}$ , if  $A \neq B$ , we have that either  $A < B$  or  $B < A$  whence  $o$  is injective.

We now see that if  $o(A) = \alpha$ , then for each  $\beta < \alpha$  we have  $\exists B < A$   $o(B) = \beta$ . For suppose this were not the case. Then we can pick  $\alpha$  and  $\beta$  contradicting the claim. Now, let  $\beta' = \min\{\gamma > \beta \mid \exists C \in \mathcal{B} \ o(C) = \gamma\}$  with  $o(B') = \beta'$ . Write  $B' = B'_k 0 \dots 0 B'_1$ . By Lemma 4.1.2,  $B'_k \neq \lambda$ , since otherwise

$$\beta \leq o(B'_{k-1} \dots 0 B'_1) + 1 = \beta',$$

contradicting the minimality of  $\beta'$ . It follows from Lemma 4.1.3 that  $\beta'$  is a limit ordinal; but this is impossible, since  $\{o(C) + 1 \mid C < B'\}$  is not unbounded in  $\beta'$ . We conclude that no such  $\beta$  exists.

Moreover, the image of  $o$  is unbounded as evidently  $o(\langle \alpha \rangle \top) \geq \alpha$ . Thus,  $o$  is an order-preserving bijection, i.e., an isomorphism.  $\square$

Lemma 4.1 above reduces the problem of computing  $o$  to that of computing  $o_1$ . In the next subsection we shall see how different  $o_\alpha$  can be related to each other.

## 4.2 Promoting and demoting worms

In this subsection we introduce an operation  $\alpha \uparrow$  that in general promotes worms to worms with higher consistency strength. As a converse operation we introduce a demoting operator  $\alpha \downarrow$ .

**Definition 4.3** ( $\alpha \uparrow$  and  $\alpha \downarrow$ ). *Let  $A$  be a worm and  $\alpha$  an ordinal. By  $\alpha \uparrow A$  we denote the worm that is obtained by simultaneously substituting each  $\beta$  that occurs in  $A$  by  $\alpha + \beta$ .*

*Likewise, if  $A \in S_\alpha$  we denote by  $\alpha \downarrow A$  the worm that is obtained by replacing simultaneously each  $\beta$  in  $A$  by  $-\alpha + \beta$ .*

Note that by Lemma 2.1, the operation  $\alpha \downarrow$  is well-defined on  $S_\alpha$ .

**Lemma 4.4.** *For  $\alpha, \beta, \gamma$  ordinals and worms  $A, B$  we have:*

1.  $\alpha \uparrow \beta < \alpha \uparrow \gamma \Leftrightarrow \beta < \gamma$ ,
2.  $\alpha \uparrow \beta \geq \beta$ ,
3.  $\alpha \downarrow (\alpha \uparrow A) = A$ ,
4.  $\alpha \uparrow (\beta \uparrow A) = (\alpha + \beta) \uparrow A$ ,

5.  $A <_\alpha B \Leftrightarrow A < B$  for  $A, B \in S_\alpha$ ,

6.  $A <_\xi B \Leftrightarrow \alpha \uparrow A <_{\alpha+\xi} \alpha \uparrow B$ .

*Proof.* The first properties are easily verified. The  $\Rightarrow$  direction of 5 is easy. The other direction follows directly from the  $\Rightarrow$  direction using irreflexivity and the fact that  $<_\alpha$  linearly orders  $\mathcal{B}_\alpha$ .

The  $\Rightarrow$  direction of 6 is the consequence of a more general observation. One can easily extend the operation  $\zeta \uparrow$  to any formula of GLP. As the operation  $\zeta \uparrow$  is order preserving on the ordinals one can easily prove by induction that any proof in GLP remains a proof after applying  $\zeta \uparrow$  to every formula appearing in it. Thus, if  $\text{GLP} \vdash \psi$ , then also  $\text{GLP} \vdash \zeta \uparrow \psi$ .

The  $\Leftarrow$  direction follows directly from the  $\Rightarrow$  direction using irreflexivity and the fact that  $<_\xi$  is a linear order on  $\mathcal{B}_\xi$ .  $\square$

We shall see that  $\alpha \uparrow$  is a well-behaved map with nice properties. In Lemma 4.7 below we prove that  $\alpha \uparrow$  can be viewed as continuous on  $S$ . In order to prove this we first need some extra machinery.

**Definition 4.5.** Let  $A$  be any worm. By  $t_\alpha(A)$  we denote the  $\alpha$ -tail of  $A$  which is the largest end segment of  $A$  that is in  $S_\alpha$ . Formally:  $t_\alpha(\lambda) := \lambda$  and

$$t_\alpha(\beta B) := \begin{cases} \beta B & \text{in case } \beta B \in S_\alpha, \\ t_\alpha(B) & \text{otherwise.} \end{cases}$$

Just as with CNFs the first component determines various properties, with worms in BNF, the tail plays an important role as is manifest in the next lemma.

**Lemma 4.6.** Let  $A$  and  $B$  be worms in BNF. We have

1.  $B < A \ \& \ A \in S_\alpha \Rightarrow t_\alpha(B) <_\alpha A$ ,

2.  $B \leq A \ \& \ B \in S_\alpha \Rightarrow B \leq_\alpha t_\alpha(A)$ .

*Proof.* For the first item we observe that we can write  $B = B't_\alpha(B)$  with the  $B'$  part possibly empty in which case clearly  $t_\alpha(B) \leq B$ . In case  $B'$  is not empty, by repeatedly applying the monotonicity axiom  $\langle \beta \rangle \psi \rightarrow \langle 0 \rangle \psi$  and transitivity in the end we conclude that  $t_\alpha(B) \leq B$ . By the assumption we thus get  $t_\alpha(B) < A$ . As both  $t_\alpha(B)$  and  $A$  are in  $S_\alpha$  we conclude  $t_\alpha(B) <_\alpha A$  by Lemma 4.4.5.

We now prove the second item. Let  $\{\beta \in A \mid \beta \leq \alpha\}$  be given in increasing order by  $\{\beta_0, \dots, \beta_n\}$ . We shall prove by induction that  $B \leq_{\beta_i} t_{\beta_i}(A)$ .

In case  $i = 0$  we see that  $t_{\beta_0}(A) = A$  and both  $B$  and  $A$  are in  $S_{\beta_0}$ . Thus we obtain  $B \leq_{\beta_0} t_{\beta_0}(A)$  directly from Lemma 4.4.5.

For the induction step we assume that  $B \leq_{\beta_i} t_{\beta_i}(A)$ . Clearly,  $t_{\beta_{i+1}}(t_{\beta_i}(A)) = t_{\beta_{i+1}}(A)$ . We suppose for a contradiction that  $t_{\beta_{i+1}}(A) <_{\beta_{i+1}} B$ . Thus, we also have  $t_{\beta_{i+1}}(A) <_{\beta_{i+1}} B$ .

Since  $A$  is in BNF, it is easy to see that  $t_{\beta_j}(A)$  is also in BNF for any  $j$ . Thus, we see that  $t_{\beta_i}(A)$  is in BNF so that that prerequisite of Lemma 3.9 is satisfied. Moreover, if  $\beta_i = \alpha$  there is no induction step to take, so we assume that

$B \in S_{\beta_i+1}$ . Thus, we can apply Lemma 3.9 to  $t_{\beta_i+1}(A) <_{\beta_i+1} B$  to obtain that  $t_{\beta_i}(A) <_{\beta_i} B$ . This contradicts irreflexivity if we use the inductive assumption that  $B \leq_{\beta_i} t_{\beta_i}(A)$ .  $\square$

We are now ready to prove that taking suprema commutes with  $\alpha\uparrow$ .

**Lemma 4.7.** *For any set  $\{A_i\}_{i \in I}$  of worms we have that*

$$\alpha\uparrow \sup_{i \in I} A_i = \sup_{i \in I} \alpha\uparrow A_i.$$

*Proof.* Without loss of generality we may assume that all the  $A_i$  are in BNF.

Note that  $\sup$  refers in both sides of the equation to the  $<$ -supremum. That is,  $D = \sup_{i \in I} C_i$  iff both  $C_i \leq D$  for all  $i \in I$  and  $D$  is the  $<$ -smallest such:  $C_i \leq E$  for all  $i \in I$  implies  $D \leq E$ .

As for all  $A_i$  we have  $A_i \leq \sup_{i \in I} A_i$ , by Lemma 4.4.6 we get for each  $i$  that

$$\alpha\uparrow A_i \leq \alpha\uparrow \sup_{i \in I} A_i,$$

whence

$$\sup_{i \in I} \alpha\uparrow A_i \leq \alpha\uparrow \sup_{i \in I} A_i.$$

To prove the other inequality we assume for a contraction that

$$\sup_{i \in I} \alpha\uparrow A_i < \alpha\uparrow \sup_{i \in I} A_i.$$

Let us write  $A$  for  $\sup_{i \in I} \alpha\uparrow A_i$ . Now, by Lemma 4.6 we see that both  $t_\alpha(A) <_\alpha \alpha\uparrow \sup_{i \in I} A_i$  and for each  $i$ ,  $\alpha\uparrow A_i \leq_\alpha t_\alpha(A)$ . By the properties from Lemma 4.4 we now see that  $\alpha\downarrow t_\alpha(A) < \sup_{i \in I} A_i$  and for each  $i$ ,  $A_i \leq \alpha\downarrow t_\alpha(A)$ . This is in contradiction with the definition of  $\sup_{i \in I} A_i$ .  $\square$

As mentioned, this lemma tells us that  $\alpha\uparrow$  can be viewed as continuous on  $S$ . Moreover, it can also be viewed as an isomorphism:

**Lemma 4.8.** *The map  $\alpha\uparrow$  is an isomorphism between  $(S, <)$  and  $(S_\alpha, <_\alpha)$ .*

*Proof.* This follows from Properties 3 and 6 of Lemma 4.4 as  $\alpha\uparrow$  is clearly a bijection.  $\square$

**Lemma 4.9.** *For  $A \in S_\alpha$  we have*

$$o_\alpha(A) = o(\alpha\downarrow A).$$

*Proof.* Immediate from Lemma 4.8.  $\square$

It is clear that Lemmata 4.1 and 4.9 together provide a complete calculus to compute  $o(A)$  for worms in  $\text{GLP}_\omega$ . For worms that contain limit ordinals we need an additional ingredient.

### 4.3 Reduction to Hyperexponentials

A key role in the larger calculus is reserved for the following functions.

**Definition 4.10** ( $e^\alpha$ ). *We define  $e^\alpha$  to be the function that enumerates  $o(S_\alpha)$ .*

We call the functions  $e^\alpha$  hyperexponentials. We shall now set out to prove various structural properties about these hyperexponentials  $e^\alpha$ . By  $o^{-1}$  we will denote the function that maps an ordinal  $\alpha$  to the unique worm  $A$  in BNF so that  $o(A) = \alpha$ . With this convention we have the following nice lemma that characterizes  $e^\alpha$  as a conjugate of the map  $\alpha \uparrow$  on worms.

**Lemma 4.11.**  *$o(S_\alpha)$  is enumerated by  $o \circ \alpha \uparrow \circ o^{-1}$ , that is,*

$$e^\alpha = o \circ \alpha \uparrow \circ o^{-1}.$$

*Proof.* In the proof we shall explicitly write  $<_0$  for the ordering on worms and  $<$  for the ordering on ordinals. As each worm  $A \in S_\alpha$  is equivalent to a worm  $A' \in \mathcal{B}_\alpha$  we see that  $o(S_\alpha) = o(\mathcal{B}_\alpha)$ . Thus, we may restrict our attention to worms in BNF. In particular, with this restriction  $o^{-1}$  is well-defined.

Lemma 4.2 told us that  $o : (\mathcal{B}, <_0) \cong (\text{Ord}, <)$ . Thus by Lemma 4.4.5 we see that for  $A, B \in S_\alpha$

$$A <_\alpha B \Leftrightarrow A <_0 B \Leftrightarrow o(A) < o(B).$$

Consequently,

$$o : (S_\alpha, <_\alpha) \cong (o(S_\alpha), <).$$

We also have by Lemma 4.8 that

$$\alpha \uparrow : (S, <_0) \cong (S_\alpha, <_\alpha).$$

Once more using that

$$o^{-1} : (\text{Ord}, <) \cong (S, <_0),$$

we see by composing the isomorphisms that

$$o \circ \alpha \uparrow \circ o^{-1} : (\text{Ord}, <) \cong (o(S_\alpha), <).$$

□

In other words, the following diagram commutes (where  $\text{Ord}$  denotes the class of ordinals):

$$\begin{array}{ccc} S & \xrightarrow{\alpha \uparrow} & S \\ \downarrow o & & \downarrow o \\ \text{Ord} & \xrightarrow{e^\alpha} & \text{Ord} \end{array}$$

So  $o$  behaves like a homomorphism between the category of worms and the category of ordinals, with morphisms of the form  $\alpha \uparrow$  and  $e^\alpha$ , respectively. If we replace worms by normal-form worms, then we in fact get an isomorphism. Let us draw some nice corollaries from our lemma.

**Corollary 4.12.**  $o(\alpha \uparrow A) = e^\alpha o(A)$

*Proof.* We may assume  $A \in \mathcal{B}$ , so  $o(\alpha \uparrow A) = o(\alpha \uparrow o^{-1}(o(A))) = e^\alpha o(A)$ .  $\square$

With this corollary we now obtain a complete calculus for computing  $o$  and  $o_\alpha$  once we know how to compute the functions  $e^\alpha$ .

**Theorem 4.13.**

1.  $o(0^n) = n$ ;
2. If  $A = A_n 0 \dots A_1 \in \mathcal{B}$  and  $A_1$  not empty, then  
 $o(A) = \omega^{o(1 \downarrow A_1)} + \dots + \omega^{o(1 \downarrow A_n)}$ ;
3.  $o(\xi \uparrow A) = e^\xi o(A)$ ;
4.  $o_\xi(A) = o(\xi \downarrow A)$  for  $A \in S_\xi$ .

Note that the last item of this lemma is redundant to compute  $o$ . It merely tells us how to reduce questions about  $o_\alpha$  to questions about  $o$ .

## 5 Computing Hyperexponentials

In this section we shall see how the functions  $e^\alpha$  can be computed. We start by observing that each  $e^\alpha$  is a well-behaved function.

**Lemma 5.1.** *Each  $e^\alpha$  is a normal function.*

*Proof.* It is clear that  $e^\alpha$  is increasing. As  $o(S_\alpha) = o(\mathcal{B}_\alpha)$ , we may assume all worms to be in BNF.

By Lemma 4.2 we know that  $o$  is an isomorphism between  $\mathcal{B}$  and  $\text{Ord}$ . In particular, both  $o$  and  $o^{-1}$  are continuous. Thus, for the continuity of  $e^\alpha$  we reason as follows.

$$\begin{aligned}
e^\alpha \sup_{i \in I} B_i &= o(\alpha \uparrow) o^{-1}(\sup_{i \in I} B_i) \\
&= o(\alpha \uparrow) \sup_{i \in I} o^{-1}(B_i) \quad \text{by Lemma 4.7} \\
&= o \sup_{i \in I} (\alpha \uparrow) o^{-1}(B_i) \\
&= \sup_{i \in I} o(\alpha \uparrow) o^{-1}(B_i) \\
&= \sup_{i \in I} e^\alpha B_i
\end{aligned}$$

$\square$

Using Lemma 4.11 we immediately get the following properties of  $e^\alpha$ .

**Lemma 5.2.**

1.  $e^0 = \text{id}$ ,
2.  $e^1 = e_0$  where  $e_0$  enumerates the set  $\{0\} \cup \{\omega^{1+\alpha} \mid \alpha \in \text{Ord}\}$ ,
3.  $e^{\alpha+\beta} = e^\alpha \circ e^\beta$ .

*Proof.* The first item is easy since  $o$  is an isomorphism (Lemma 4.2).

We now see that  $e^1\alpha = e_0\alpha$ . The case  $\alpha = 0$  is trivial. For non-empty worms  $A \in \mathcal{B}$  we have

$$\begin{aligned} o \circ (1 \uparrow)A &= \omega^{o_1((1 \uparrow)A)} && \text{by Lemma 4.1.3} \\ &= \omega^{o(A)} && \text{by Lemma 4.9.} \end{aligned}$$

Thus, using Lemma 4.11 we see that for  $\alpha > 0$  we have that

$$e^1\alpha = o \circ (1 \uparrow) \circ o^{-1}\alpha = \omega^{o(o^{-1}(\alpha))} = \omega^\alpha = e_0\alpha.$$

For the last item we see that

$$\begin{aligned} e^\alpha \circ e^\beta &= o \circ \alpha \uparrow \circ o^{-1} \circ o \circ \beta \uparrow \circ o^{-1} \\ &= o \circ \alpha \uparrow \circ \beta \uparrow \circ o^{-1} \\ &= o \circ (\alpha + \beta) \uparrow \circ o^{-1}. \end{aligned}$$

□

These three properties as stated in Lemma 5.2 already fix quite some properties of the function  $e^\alpha$ . In [5] the authors study abstractly any progression of normal functions that satisfy the three properties of Lemma 5.2 for arbitrary normal functions  $e_0$  and call these progressions *weak hyperations* (see Definition 6.1).

Clearly, these three properties say nothing about the behavior of  $e^\alpha$  for additively indecomposable  $\alpha$ . To deal with those ordinals we have the following lemma.

**Lemma 5.3.** *Let  $\lambda$  be an additively indecomposable limit ordinal. We have that*

$$e^\lambda(\beta + 1) = \cup_{\lambda' < \lambda} e^{\lambda'}(e^\lambda(\beta) + 1).$$

*Proof.*  $e^\lambda(\beta + 1)$  is the  $(\beta + 1)$ -th element of  $o(\mathcal{B}_\lambda)$ . With a reasoning similar to the proof of Lemma 4.1.2, it is easy to see that  $e^\lambda(\beta + 1) = o(\lambda B)$  for some  $B \in \mathcal{B}_\lambda$  so that

$$o(B) = e^\lambda(\beta). \tag{1}$$

As  $\lambda$  is additionally indecomposable, we see that  $\lambda' \downarrow \alpha = \alpha$  for any  $\lambda' < \lambda$  and any  $\alpha \geq \lambda$ . In particular we see that

$$\lambda' \downarrow B = B \quad \text{for any } \lambda' < \lambda. \tag{2}$$

We claim that

$$o(\lambda B) = \cup_{\lambda' < \lambda} o(\lambda' B). \quad (3)$$

Once we have proven the claim, it is easy to see that the main result follows.

$$\begin{aligned} e^\lambda(\beta + 1) &= o(\lambda B) && \text{by (3)} \\ &= \cup_{\lambda' < \lambda} o(\lambda' B) && \text{by Corollary 4.12} \\ &= \cup_{\lambda' < \lambda} e^{\lambda'} o(0\lambda' \downarrow B) && \text{by Lemma 4.1.2} \\ &= \cup_{\lambda' < \lambda} e^{\lambda'} (o(\lambda' \downarrow B) + 1) && \text{by (2)} \\ &= \cup_{\lambda' < \lambda} e^{\lambda'} (o(B) + 1) && \text{by (1)} \\ &= \cup_{\lambda' < \lambda} e^{\lambda'} (e^\lambda(\beta) + 1) \end{aligned}$$

To conclude the proof we should address the claim as formulated in (3). Note that  $o(\lambda B)$  is the ordertype of all worms  $<_0$ -below  $\lambda B$ . Thus, to prove our claim, it suffices to see that for each  $C$  with  $B \leq C < \lambda B$ , there is a  $\lambda' < \lambda$  with  $C < \lambda' B$ . We may assume  $C$  to be in BNF. If we apply Lemma 4.6 to  $B \leq C < \lambda B$  we get that

$$B \leq_\lambda t_\lambda C <_\lambda \lambda B.$$

Consequently,  $C$  is of the form  $C'B$  with all ordinals in  $C'$  strictly below  $\lambda$ . If we take  $\lambda' := \max\{\gamma \mid \gamma \in C'\} + 1$  then certainly  $\lambda' B > C'B (= C)$  and the claim is proved.  $\square$

Now that we have proved this lemma we finally have fully determined all functions  $e^\alpha$ .

**Theorem 5.4.** *For ordinals  $\alpha$  and  $\beta$ , the values  $e^\alpha(\beta)$  are determined by the following recursion.*

1.  $e^\alpha 0 = 0$  for all  $\alpha \in \text{Ord}$ ;
2.  $e^1 = e_0$  where  $e_0$  enumerates the set  $\{0\} \cup \{\omega^{1+\alpha} \mid \alpha \in \text{Ord}\}$ ;
3.  $e^{\alpha+\beta} = e^\alpha e^\beta$ ;
4.  $e^\alpha(\lambda) = \cup_{\beta < \lambda} e^\alpha(\beta)$  for limit ordinals  $\lambda$ ;
5.  $e^\lambda(\beta + 1) = \cup_{\lambda' < \lambda} e^{\lambda'}(e^\lambda(\beta) + 1)$  for  $\lambda$  an additively indecomposable limit ordinal.

## 6 Hyperations and Veblen progressions

In this section we shall see how the hyperexponentials can be related to the well-studied Veblen progressions.

## 6.1 Hyperations

*Hyperation* is a form of transfinite iteration of normal functions which is introduced and systematically studied in [5]. It is based on the additivity of finite iterations, that is  $f^{m+n} = f^m \circ f^n$ , generalizing this to the transfinite setting.

**Definition 6.1** (Weak hyperation). *A weak hyperation of a normal function  $f$  is a family of normal functions  $\langle g^\xi \rangle_{\xi \in \text{On}}$  such that*

1.  $g^0 \xi = \xi$  for all  $\xi$ ,
2.  $g^1 = f$ ,
3.  $g^{\xi+\zeta} = g^\xi g^\zeta$ .

Par abuse de langage we will often write just  $g^\xi$  instead of  $\langle g^\xi \rangle_{\xi \in \text{On}}$ . Weak hyperations are not unique. However, if we impose a minimality condition, one can prove (see [5]) that there is a unique minimal hyperation.

**Definition 6.2** (Hyperation). *A weak hyperation  $g^\xi$  of  $f$  is minimal if it has the property that, whenever  $h^\xi$  is a weak hyperation of  $f$  and  $\xi, \zeta$  are ordinals, then  $g^\xi \zeta \leq h^\xi \zeta$ .*

*If  $f$  has a (unique) minimal weak hyperation, we call it the hyperation of  $f$  and denote it  $f^\xi$ .*

Note that we have already proven that  $e^\alpha$  is a weak hyperation. We shall now show its minimality.

**Theorem 6.3.**  *$e^\alpha$  is the minimal collection of normal functions satisfying*

1.  $e^0 = \text{id}$ ,
2.  $e^1 = e_0$ ,
3.  $e^{\alpha+\beta} = e^\alpha \circ e^\beta$ .

*Proof.* Suppose that  $\{f^\alpha\}_{\alpha \in \text{Ord}}$  is a collection of normal functions such that 1.–3. holds. By induction on  $\alpha$  we shall see that  $e^\alpha(\beta) \leq f^\alpha(\beta)$ .

For  $\alpha = 0$  and  $\alpha = 1$  this is obvious and for additively decomposable ordinals we see that

$$e^{\alpha+\beta} = e^\alpha e^\beta \leq_{\text{IH}} f^\alpha f^\beta = f^{\alpha+\beta}.$$

So, let  $\alpha$  be an indecomposable limit ordinal. Assume for a contradiction that we can pick the minimal such  $\alpha$  and the corresponding minimal  $\beta'$  such that  $f^\alpha(\beta') < e^\alpha(\beta')$ . Clearly  $\beta'$  cannot be 0. But as both  $f^\alpha$  and  $e^\alpha$  are continuous,  $\beta'$  can also not be a limit ordinal.

Thus we conclude  $\beta' = \beta + 1$  for some ordinal  $\beta$ . As  $\alpha$  is an indecomposable limit ordinal we can apply Lemma 5.3 to see that

$$e^\alpha(\beta + 1) = \cup_{\alpha' < \alpha} e^{\alpha'}(e^\alpha(\beta) + 1) \leq \cup_{\alpha' < \alpha} f^{\alpha'}(f^\alpha(\beta) + 1). \quad (\dagger)$$



As for  $\alpha' < \alpha$  we have  $\alpha' + \alpha = \alpha$ , by Property 3, we see that

$$f^\alpha(\beta + 1) = \cup_{\alpha' < \alpha} f^{\alpha'} f^\alpha(\beta + 1).$$

But, as  $f^\alpha$  is monotone we also see that  $f^\alpha(\beta + 1) \geq f^\alpha(\beta) + 1$  whence by monotonicity of all of the  $f^{\alpha'}$  we see that

$$f^\alpha(\beta + 1) = \cup_{\alpha' < \alpha} f^{\alpha'} f^\alpha(\beta + 1) \geq \cup_{\alpha' < \alpha} f^{\alpha'}(f^\alpha(\beta) + 1).$$

We combine this with  $(\dagger)$  to conclude that

$$e^\alpha(\beta + 1) \leq f^\alpha(\beta + 1)$$

which contradicts our assumption.  $\square$

With this theorem we have shown that the collection of hyperexponentials  $e^\alpha$  is the unique hyperation of  $e_0$ .

## 6.2 Veblen progressions

It is not hard to see that each normal function has an unbounded class of fixpoints. For example the first fixpoint of the function  $\varphi_0 : x \mapsto \omega^x$  is

$$\sup\{\omega, \omega^\omega, \omega^{\omega^\omega}, \dots\}$$

and is denoted  $\varepsilon_0$ . In his seminal paper [11], Veblen considered for each normal function  $f$  its derivative  $f'$  that enumerates the fixpoints of  $f$ . This process can be transfinitely continued by defining  $f_\alpha$  to be the ordinal function that enumerates those ordinals which are simultaneous fixpoints of all  $f_\beta$  for  $\beta < \alpha$ . In particular, the  $\varphi_\alpha$  is the thus obtained Veblen progression by starting with  $\varphi_0 : x \mapsto \omega^x$ .

Beklemishev noted in [3] that in the setting of GLP it is desirable to start with a slightly different function:  $\hat{\varphi}_0 : x \mapsto \omega^{1+x}$ .

In [5] and in this paper the authors realized that, moreover it is desirable to have 0 in the range of the initial function (see also Lemma 6.6 of this paper), whence we started from  $e_0$  which enumerates the set

$$\{0\} \cup \{\omega^{1+\alpha} \mid \alpha \in \text{On}\}.$$

We shall denote the corresponding Veblen progression by  $e_\alpha$ . In general, if  $f$  is some normal function, we shall denote by  $f_\alpha$  the Veblen progression based on  $f_0 = f$ . Note that, if  $\alpha < \beta$ , we have that  $f_\beta(\gamma)$  is always a fixpoint of  $f_\alpha$ , i.e.,  $f_\beta = f_\alpha \circ f_\beta$ .

One readily observes that

$$\begin{aligned} e_\alpha(0) &= 0 && \text{for all } \alpha; \\ e_0(1 + \beta) &= \varphi_0(1 + \beta) = \hat{\varphi}_0(\beta) && \text{for all } \beta; \\ e_{1+\alpha}(1 + \beta) &= \varphi_{1+\alpha}(\beta) = \hat{\varphi}_{1+\alpha}(\beta) && \text{for all } \alpha, \beta. \end{aligned}$$

### 6.3 Veblen progressions and hyperations

In [5] the authors established a close connection between hyperations and Veblen progressions. In particular, any hyperation can be seen as a natural refinement of a Veblen progression, and vice-versa, any Veblen progression can be refined to a hyperation as expressed in the following two theorems.

**Theorem 6.4.** *Let  $f$  be a normal function and let  $f_\alpha$  be the Veblen progression based on it. Given an ordinal  $\alpha$ , we have that  $f^{\omega^\alpha} = f_\alpha$ .*

**Theorem 6.5.** *Let  $g^\xi$  be a weak hyperation of a normal function  $f$ . If we moreover have that  $g^{\omega^\alpha} = f_\alpha$  for each  $\alpha$  then  $g^\xi = f^\xi$ .*

Now that we know that  $e^\alpha$  is the unique hyperation of  $e_0$ , we note that Theorem 6.4 together with the additivity ( $e^{\alpha+\beta} = e^\alpha e^\beta$ ) yields a reduction of computing  $e^\alpha$  to the better known Veblen-like functions  $e_\alpha$ . For if  $\alpha = \omega^{\alpha_1} + \dots + \omega^{\alpha_n}$ , then

$$e^\alpha = e_{\alpha_1} \circ \dots \circ e_{\alpha_n}.$$

The calculus for  $o$  that was presented in [3] was actually based on this equation in a slightly different presentation. However, we do not need the full theory of hyperations to obtain Theorem 6.4 for hyperexponentiations:

**Lemma 6.6.**  $e^{\omega^\alpha} = e_\alpha$ .

*Proof.* Our proof proceeds by induction on  $\alpha$  and closely follows the proof of Theorem 2 of [3].

Recall that the  $E_\alpha$  denote the images of the corresponding Veblen progression  $e_\alpha$  and  $E_0 = \{0\} \cup \{\omega^{1+\alpha} \mid \alpha \in \text{On}\}$ . We shall prove an equivalent statement that  $o(S_{\omega^\alpha}) = E_\alpha$  by induction on  $\alpha$ . For  $\alpha = 0$  this is just the second item of Lemma 5.2.

First we remark that  $o(\cap_i S_{\alpha_i}) = \cap_i o(S_{\alpha_i})$ . The  $\subseteq$  direction is immediate. For the other inclusion we assume that  $\beta \in \cap_j o(S_{\alpha_j})$  and observe that if  $\beta \in o(S_{\alpha_i})$ , then there is some  $A_i \in S_{\alpha_i}$  with  $o(A_i) = \beta$ . But then there is also some  $A'_i \in S_{\alpha_i} \cap \text{BNF}$  equivalent to  $A_i$  with  $o(A'_i) = \beta$ . As BNFs are unique we see that  $A'_i \in \cap_j S_{\alpha_j}$ .

We now consider successor ordinals.

$$\begin{aligned} o(S_{\omega^{\alpha+1}}) &= o(S_{\omega^\alpha \cdot \omega}) \\ &= o(\cap_{n < \omega} S_{\omega^\alpha \cdot n}) \\ &= \cap_{n < \omega} o(S_{\omega^\alpha \cdot n}) \end{aligned}$$

We conclude by showing that  $\cap_{n < \omega} o(S_{\omega^\alpha \cdot n}) = E_{\alpha+1}$ . By the induction hypothesis we have  $o(S_{\omega^\alpha}) = E_\alpha$  so that by Lemma 5.2.3 we see that  $o(S_{\omega^\alpha \cdot n})$  is enumerated by  $(e^{\omega^\alpha})^n = e_\alpha^n$ , that is,  $n$  iterations<sup>1</sup> of  $e_\alpha$ . Thus clearly if  $\beta$  is a fixed point of  $e_\alpha$ , that is,  $\beta \in E_{\alpha+1}$ , then for any  $n < \omega$   $e_\alpha^n(\beta) = \beta$ , and  $\beta \in \cap_{n < \omega} o(S_{\omega^\alpha \cdot n})$ .

---

<sup>1</sup>Note, we do not have in general that  $(e_\alpha)^n = (e^n)_\alpha$ . When we write  $e_\alpha^n$  here we shall mean  $(e_\alpha)^n$ .

To see that also  $\cap_{n < \omega} o(S_{\omega^\alpha \cdot n}) \subseteq E_{\alpha+1}$  we consider some  $\beta \in \cap_{n < \omega} o(S_{\omega^\alpha \cdot n})$ . Thus we obtain a sequence  $\beta_1 \geq \beta_2 \geq \dots$  with for each  $n$ ,  $e_\alpha^n(\beta_n) = \beta$ . By well-foundedness, for some  $k$  we have that  $\beta_{k+1} = \beta_k$ . We reason as in [3] to see that

$$\beta = e_\alpha^{n+1}(\beta_{k+1}) = e_\alpha^{n+1}(\beta_k) = e_\alpha(e_\alpha^n(\beta_k)) = e_\alpha(\beta).$$

Thus  $\beta$  is a fixed point of  $e_\alpha$  and  $\beta \in E_{\alpha+1}$ .

For limit ordinals  $\alpha$  we observe that

$$o(S_{\omega^\alpha}) = o(\cap_{\beta < \alpha} S_{\omega^\beta}) = \cap_{\beta < \alpha} o(S_{\omega^\beta}) =_{\text{IH}} \cap_{\beta < \alpha} E_\beta = E_\alpha.$$

□

## References

- [1] L. D. Beklemishev, D. Fernández-Duque, and J. J. Joosten. On provability logics with linearly ordered modalities. <http://arxiv.org/abs/1210.4809>, 2012.
- [2] L.D. Beklemishev. Provability algebras and proof-theoretic ordinals, I. *Annals of Pure and Applied Logic*, 128:103–124, 2004.
- [3] L.D. Beklemishev. Veblen hierarchy in the context of provability algebras. In P. Hájek, L. Valdés-Villanueva, and D. Westerstahl, editors, *Logic, Methodology and Philosophy of Science, Proceedings of the Twelfth International Congress*. Kings College Publications, 2005.
- [4] L.D. Beklemishev. The worm principle. In Z. Chatzidakis, P. Koepke, and W. Pohlers, editors, *Logic Colloquium 2002, Lecture Notes in Logic 27*. ASL Publications, 2006.
- [5] Fernández-Duque, D. and Joosten, J. J. Hyperations, Veblen progressions and transfinite iteration of ordinal functions. <http://arxiv.org/abs/1205.2036>, 2012.
- [6] Fernández-Duque, D. and Joosten, J. J. Turing progressions and their well-orders. In *How the world computes*, Lecture Notes in Computer Science, pages 212–221. Springer, 2012.
- [7] Fernández-Duque, D. and Joosten, J. J. Well-orders in the transfinite Japaridze algebra II: Turing progressions and their well-orders. <http://arxiv.org/abs/1204.4743>, 2012.
- [8] K. N. Ignatiev. On strong provability predicates and the associated modal logics. *The Journal of Symbolic Logic*, 58:249–290, 1993.
- [9] G. Japaridze. The polymodal provability logic. In *Intensional logics and logical structure of theories: material from the Fourth Soviet-Finnish Symposium on Logic*. Metsniereba, Telavi, 1988. In Russian.

- [10] W. Pohlers. *Proof Theory, The First Step into Impredicativity*. Springer-Verlag, Berlin Heidelberg, 2009.
- [11] O. Veblen. Continuous increasing functions of finite and transfinite ordinals. *Transactions of the American Mathematical Society*, 9:280–292, 1908.